Journal of Statistical Physics, Vol. 21, No. 3, 1979

The Kinetic Ising Model: Exact Susceptibilities of Two Simple Examples

J. C. Kimball[1,2]

Received January 30, 1979; revised March 13, 1979

The susceptibility of a modified version of the one-dimensional kinetic Ising model is obtained and compared with the susceptibility of the Glauber version of this model. Spin-flip rates in the new model are picked so no spin-flip rate vanishes as the temperature vanishes. Despite the more rapid spin flips, the new model exhibits an infinitely slow approach to equilibrium in the low-temperature limit which is similar to the slowing down exhibited in the Glauber model. The new model also exhibits two different decay rates toward equilibrium, which are called the transient and slow decay rates. The Glauber model is characterized by only a single decay rate toward equilibrium.

KEY WORDS: Glauber model; master equation; frequency-dependent susceptibility; Kubo formula; transient response.

1. INTRODUCTION

The one-dimensional kinetic Ising model was introduced by Glauber[1] in 1963 as a soluble problem in nonequilibrium statistical mechanics (however, see also Ref. 2). Properties of the Glauber model and other time-dependent generalizations of the Ising model have been reviewed by Kawasaki[3] and by Binder.[4]

One is naturally motivated to consider generalizations of Glauber's original model. Works on more complicated models have included the extension to higher dimensions, the application of magnetic fields, and the choice of alternative spin-flip rates.[5-9] Unfortunately, all nontrivial generalizations presented so far have been solved only approximately.[10-17]

Work supported in part by National Science Foundation grant DMR 78-03408.

[1] Department of Physics, University of California, Berkeley, California.

[2] Permanent address: Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana.

The major contribution of this paper is the presentation of a modification of Glauber's original model, for which the uniform field susceptibility may be obtained exactly. The modification is an alternative choice of the spin-flip rates, which changes the dynamics of the system but leaves the equilibrium properties unchanged. Only the linear response of this new version of the kinetic Ising model will be investigated.

Like the Glauber solution, the new version of the kinetic Ising model exhibits an infinitely slow relaxation to equilibrium as the temperature is lowered to zero. The slowing occurs even though no spin-flip rate becomes slow in the new model. Also, the new model exhibits both "slow" and "transient" responses to an applied magnetic field, while the Glauber solutions can be characterized by a single time scale.

The general structure of the kinetic Ising model and the formalism necessary to obtain the linear response are discussed in Section 2. Some of the more tedious algebra of this section has been placed in the appendix. The formal solutions for the susceptibility of the Glauber model and the new version of the kinetic Ising model are presented in Section 3. A qualitative discussion of the nature of these two solutions, their similarities and differences, is presented in Section 4.

2. THE MASTER EQUATION AND LINEAR RESPONSE

The time dependence of the kinetic Ising model is described by a master equation. It is convenient to change variables so that the master equation can be written in terms of a Hermitian operator called T. Then the linear response of the system to an applied magnetic field can be written as a simple expression which is analogous to a classical Kubo formula. A description of the Ising model, the master equation and T, and the Kubo formula follows.

The energy of a configuration α of an Ising model is

$$E(\alpha) = -J \sum_{n=1}^{N} \sigma_n \sigma_{n+1} - \sum_{n=1}^{N} H\sigma_n \tag{2.1}$$

where the sign of each σ_n is prescribed by the configuration α. The nearest neighbor spin–spin interaction J is assumed to be positive, and the magnetic field is H. In thermal equilibrium, the probability that the Ising model is in configuration α is

$$P_\alpha = \exp[-\beta E(\alpha)]/Z \tag{2.2}$$

where β is the inverse temperature in units of Boltzmann's constant, and Z is the partition function.

Nonequilibrium situations are of interest when one considers the kinetic Ising model. In this case, the probability of finding the system in configuration

α is a time-dependent quantity $P_\alpha(t)$, which is not necessarily given by the equilibrium probability P_α. The time dependence of the probabilities is described by a master equation

$$\frac{dP_\alpha(t)}{dt} = -\sum_\gamma [W(\gamma|\alpha)P_\alpha(t) - W(\alpha|\gamma)P_\gamma(t)] \tag{2.3}$$

Here, $W(\gamma|\alpha)$ is the rate at which an ensemble of systems in configuration α would make transitions to configuration γ. In equilibrium, the rate of transitions from α to γ must cancel the transitions from γ to α. This condition of detailed balance means

$$W(\gamma|\alpha)P_\alpha = W(\alpha|\gamma)P_\gamma \tag{2.4}$$

The master equation and the probabilities $P_\alpha(t)$ have a simple intuitive meaning, but from a formal point of view it is more convenient to work with a new set of variables $\psi_\alpha(t)$, which give the master equation a manifestly Hermitian form.[3] The new variables are

$$\psi_\alpha(t) = P_\alpha(t)/\sqrt{P_\alpha} \tag{2.5}$$

The notation can be made more compact by treating each configuration of the spin system as one of the orthonormal basis vectors of a 2^N-dimensional vector space, so $\alpha \to |\alpha\rangle$. Using this vector space notation and the new variables, we find that the master equation becomes

$$d|\psi(t)\rangle/dt = -T|\psi(t)\rangle \tag{2.6}$$

where

$$|\psi(t)\rangle = \sum_\alpha \psi_\alpha(t)|\alpha\rangle \tag{2.7}$$

and the Hermitian operator T is defined in terms of its matrix elements.

$$\begin{aligned}\langle\alpha|T|\gamma\rangle &= -[W(\alpha|\gamma)W(\gamma|\alpha)]^{1/2}; \qquad \alpha \neq \gamma \\ \langle\alpha|T|\alpha\rangle &= +\sum_{\gamma\neq\alpha} W(\gamma|\alpha)\end{aligned} \tag{2.8}$$

The time-independent equilibrium solution to the master equation is denoted $|\psi\rangle$, with the time index eliminated, and from Eq. (2.6),

$$T|\psi\rangle = 0 \tag{2.9}$$

Note that T has a nonnegative spectrum, because no probability can grow with time. This means $|\psi\rangle$ is the "ground state" of T.

Equilibrium properties of the Ising model can be described in terms of the ground-state vector $|\psi\rangle$ and the Pauli matrices $\sigma_n{}^z$ associated with the spin variables σ_n. For example,

$$\langle\sigma_l\sigma_n\rangle = \langle\psi|\sigma_l{}^z\sigma_n{}^z|\psi\rangle \tag{2.10}$$

[3] This notation is similar to that introduced in Ref. 18.

The linear response of the kinetic Ising model can be expressed in terms of a Kubo formula.[19] If a small magnetic field "pulse" is applied to lattice site n at time $t = 0$, the resulting magnetization at lattice site l at time $t \geqslant 0$ is proportional to the reponse function

$$\phi_{nl}(t) = \beta\langle\sigma_l(t)\dot{\sigma}_n(0)\rangle \tag{2.11}$$

where $\dot{\sigma}_n(0)$ is the time derivative of the spin at site n, and $\sigma_l(t)$ is the spin at site l a time t later. Because the operator T describes the time evolution of the system, one can argue that

$$\sigma_l^z(t) \approx e^{tT}\sigma_l^z e^{-tT}, \qquad \dot{\sigma}_n^z \approx [T, \sigma_n^z] \tag{2.12}$$

Combining the results of Eqs. (2.9)–(2.12) means that the response function can be written in terms of T and $|\psi\rangle$,

$$\phi_{nl}(t) = \beta\langle\psi|\sigma_l^z e^{-tT} T\sigma_n^z|\psi\rangle \tag{2.13}$$

Clearly the arguments presented here leading to the response function $\phi_{nl}(t)$ are only suggestive. A rigorous derivation of Eq. (2.13) is essentially contained in Kawasaki's review article.[3]

The susceptibility of the kinetic Ising model, for the case of a uniform field, can be expressed in terms of the magnetization operator

$$M = \sum_n \sigma_n^z \tag{2.14}$$

The uniform field response function is

$$\phi(t) = \beta\langle\psi|Me^{-tT}TM|\psi\rangle \tag{2.15}$$

and the susceptibility is the positive time Fourier transform of the response function

$$\chi(\omega) = \int_0^\infty e^{i\omega t}\phi(t)\,dt = \beta\langle\psi|M\left(\frac{T}{T - i\omega}\right)M|\psi\rangle \tag{2.16}$$

Ordinarily, one thinks of $\chi(\omega)$ as the experimentally measurable quantity, but sometimes the response function $\phi(t)$ gives one more physical intuition.

3. THE SOLUTIONS

The time-dependent linear response of the one-dimensional kinetic Ising model depends in detail on the spin-flip rates $W(\alpha|\gamma)$, since these rates determine the structure of T. The two solutions presented here correspond to cases where the $W(\alpha|\gamma)$ are picked to make T particularly simple. The spin-flip rates and some of the formal manipulations necessary to cast T into a convenient form are presented in Section 3.1. The solutions to the Glauber model and the modified model are presented in Sections 3.2 and 3.3.

3.1. General Structure

The transition rates for the one-dimensional kinetic Ising model can be expressed in terms of two constants, W_0 and $\overline{W}$.

The spin-flip rate for a spin between two antiparallel spins is W_0. The rate for a spin between parallel spins is $\overline{W}e^{2\beta J}$ if the spin is flipping toward alignment with its nearest neighbors, and it is $\overline{W}e^{-2\beta J}$ otherwise. These rates are shown pictorially as:

$$
\begin{aligned}
(\uparrow, \uparrow, \downarrow) &\leftrightarrow (\uparrow, \downarrow, \downarrow) \approx W_0 \\
(\uparrow, \downarrow, \uparrow) &\rightarrow (\uparrow, \uparrow, \uparrow) \approx \overline{W}e^{2\beta J} \\
(\uparrow, \uparrow, \uparrow) &\rightarrow (\uparrow, \downarrow, \uparrow) \approx \overline{W}e^{-2\beta J}
\end{aligned}
\tag{3.1}
$$

It is important to note that there is no simple physical argument which relates W_0 and $\overline{W}$. The only difference between Glauber's form of the kinetic Ising model and the modification presented here lies in the choice of these two parameters.

Once the spin-flip rates are determined, the operator T can be obtained. It is convenient to write T in terms of Pauli matrices. The nondiagonal part of T is T_{ND},

$$T_{\mathrm{ND}} = -\tfrac{1}{2} \sum_n \sigma_n{}^x [W_0(1 - \sigma_{n-1}^z \sigma_{n+1}^z) + \overline{W}(1 + \sigma_{n-1}^z \sigma_{n+1}^z)] \tag{3.2}$$

Here $(1 \pm \sigma_{n-1}^z \sigma_{n+1}^z)/2$ projects out the configurations in which the spins at sites $(n - 1)$ and $(n + 1)$ are parallel or antiparallel, and $\sigma_n{}^x$ flips the spin at site n. A similar construction yields the diagonal part of T,

$$
\begin{aligned}
T_{\mathrm{D}} = {} & \tfrac{1}{2} W_0 \sum_n (1 - \sigma_{n-1}^z \sigma_{n+1}^z) \\
& + \tfrac{1}{4} \overline{W} e^{+2\beta J} \sum_n (1 + \sigma_{n-1}^z \sigma_{n+1}^z)(1 - \sigma_n{}^z \sigma_{n+1}^z) \\
& + \tfrac{1}{4} \overline{W} e^{-2\beta J} \sum_n (1 + \sigma_{n-1}^z \sigma_{n+1}^z)(1 + \sigma_n{}^z \sigma_{n+1}^z)
\end{aligned}
\tag{3.3}
$$

This form for T $(= T_{\mathrm{ND}} + T_{\mathrm{D}})$ is cumbersome. The algebra presented in the appendix leads to a decomposition of T into "single-site" terms T_n, which have the relatively simple properties represented in the following five equations:

$$T = \sum_n T_n \tag{3.4}$$

with

$$T_n |\psi\rangle = 0 \tag{3.5}$$

For $n \neq m$,

$$[T_n, \sigma_m{}^z] = 0 \tag{3.6}$$

Also,

$$T_n \sigma_n{}^z |\psi\rangle = [a\sigma_n{}^z + b(\sigma_{n-1}^z + \sigma_{n+1}^z) + c\sigma_{n-1}^z \sigma_n{}^z \sigma_{n+1}^z] |\psi\rangle \tag{3.7}$$

with

$$a = W_0 + \overline{W}\cosh(2\beta J), \qquad b = -\overline{W}\sinh(2\beta J)$$
$$c = -W_0 + \overline{W}\cosh(2\beta J) \tag{3.8}$$

3.2. The Glauber Model

The dynamics of the kinetic Ising model is particularly simple if one chooses the spin-flip rates suggested by Glauber. If $(-\sigma_n \leftarrow \sigma_n)$ denotes the transition in which only σ_n changes sign, the simple choice is

$$W(-\sigma_n \leftarrow \sigma_n) = W_0\{1 - \sigma_n \tanh[\beta J(\sigma_{n-1} + \sigma_{n+1})]\} \tag{3.9}$$

If W_0 is independent of temperature, this corresponds to the choice of parameters described in Section 3.1 as

$$W_0 = \text{const}, \qquad \overline{W} = W_0/\cosh(2\beta J) \tag{3.10}$$

With this choice of the spin-flip parameters, the coefficient c in Eqs. (3.7) and (3.8) vanishes. As a result, $M|\psi\rangle$ is an eigenfunction of T [M is defined in Eq. (2.14)]

$$TM|\psi\rangle = \lambda_G M|\psi\rangle \tag{3.11}$$

with

$$\lambda_G = 2W_0[1 - \tanh(2\beta J)]$$

The response function [Eq. (2.15)] is then

$$\phi(t) = \beta\langle\psi|MM|\psi\rangle\lambda_G \exp(-\lambda_G t) \tag{3.12}$$

so

$$\phi(t) = \beta N[\exp(2\beta J)]\lambda_G \exp(-\lambda_G t) \tag{3.13}$$

The frequency-dependent susceptibility is then obtained from Eq. (2.16),

$$\chi(\omega) = \beta N \exp(2\beta J)/(1 - i\omega/\lambda_G) \tag{3.14}$$

This gives the traditional Ising model susceptibility when $\omega = 0$.

3.3. The New Solution

The problem with any choice of the spin-flip parameters (W_0 and $\overline{W}$) other than Glauber's choice is that $M|\psi\rangle$ will no longer be an eigenfunction of T. The alternative solution to the kinetic Ising model which is presented here is simply another judicious choice of the spin-flip parameters, so that $\sigma^z|\psi\rangle$ is the sum of two eigenfunctions of T. Instead of forcing c to be zero, the parameters W_0 and $\overline{W}$ are chosen so that

$$b + c = 0 \tag{3.15}$$

with b and c still given by Eq. (3.8). This is equivalent to the requirement that

$$\overline{W} = W_0 e^{2\beta J} \tag{3.16}$$

or, pictorially,

$$\begin{aligned} (\uparrow, \uparrow, \downarrow) &\leftrightarrow (\uparrow, \downarrow, \downarrow) \approx W_0 \\ (\uparrow, \downarrow, \uparrow) &\rightarrow (\uparrow, \uparrow, \uparrow) \approx W_0 e^{4\beta J} \\ (\uparrow, \uparrow, \uparrow) &\rightarrow (\uparrow, \downarrow, \uparrow) \approx W_0 \end{aligned} \tag{3.17}$$

The linear response of this system is obtained by writing $M|\psi\rangle$ in terms of the eigenfunctions of T. The terms of Eq. (3.8) become

$$c = W_0 e^{2\beta J} \sinh(2\beta J), \qquad b = -c, \qquad a = c + 2 \tag{3.18}$$

Let

$$S = \sum_n \sigma_{n-1}^z \sigma_n^z \sigma_{n+1}^z \tag{3.19}$$

Then, applying the properties of T described in Eqs. (3.4)–(3.8), we obtain

$$TM|\psi\rangle = (a - 2c)M|\psi\rangle + cS|\psi\rangle \tag{3.20}$$

and

$$TS|\psi\rangle = 3aS|\psi\rangle - 3cM|\psi\rangle \tag{3.21}$$

These two equations (3.20) and (3.21) form a closed system. No higher order terms appear if $b + c = 0$. Thus $M|\psi\rangle$ can be written in terms of two eigenvectors of T, denoted $|\Omega_+\rangle$ and $|\Omega_-\rangle$. The eigenvectors are written as

$$|\Omega_\pm\rangle = \alpha_\pm M|\psi\rangle + \beta_\pm S|\psi\rangle \tag{3.22}$$

and the associated eigenvalues are $\lambda_\pm$. The equations which determine the eigenvectors can be written as a 2×2 matrix equation

$$\begin{pmatrix} a - 2c & -3c \\ c & 3a \end{pmatrix} \begin{pmatrix} \alpha_\pm \\ \beta_\pm \end{pmatrix} = \lambda_\pm \begin{pmatrix} \alpha_\pm \\ \beta_\pm \end{pmatrix} \tag{3.23}$$

The matrix does not appear Hermitian because $M|\psi\rangle$ and $S|\psi\rangle$ are not orthogonal or normalized to unity. Of course $\langle\Omega_+/\Omega_-\rangle = 0$. The eigenvalues of the matrix are

$$\lambda_\pm = c + 4 \pm r \tag{3.24}$$

where

$$r = (c^2 + 8c + 4)^{1/2} \tag{3.25}$$

The normalization of the eigenvectors $|\Omega_\pm\rangle$ is picked so that $\alpha_+ + \alpha_- = 1$ and $\beta_+ + \beta_- = 0$. This means

$$M|\psi\rangle = |\Omega_+\rangle + |\Omega_-\rangle \tag{3.26}$$

and the eigenvector coefficients are

$$\begin{pmatrix} \alpha_\pm \\ \beta_\pm \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \mp (c + 1)/r \\ \pm c/(2r) \end{pmatrix} \tag{3.27}$$

Having found the relevant eigenvectors and eigenvalues of T, we obtain the response function $\phi(t)$ from Eq. (2.15) and the decomposition of $M|\psi\rangle$ given by Eq. (3.26),

$$\phi(t) = \beta[\mathcal{O}_+\lambda_+ \exp(-\lambda_+ t) + \mathcal{O}_-\lambda_- \exp(-\lambda_- t)] \tag{3.28}$$

where

$$\mathcal{O}_\pm = \langle\psi|M|\Omega_\pm\rangle = \frac{1-\eta^2}{2r}\left(\frac{r \mp (2+c)}{(1-\eta)^2} \mp c\right) \tag{3.29}$$

and $\eta = \tanh(\beta J)$. The form for $\mathcal{O}_\pm$ is obtained from equilibrium spin–spin correlation functions. The susceptibility is obtained in the same way as was done for the Glauber model, and

$$\chi(\omega) = \beta N\left(\frac{\mathcal{O}_+}{1 - i\omega/\lambda_+} + \frac{\mathcal{O}_-}{1 - i\omega/\lambda_-}\right) \tag{3.30}$$

4. DISCUSSION

The linear response properties of the two versions of the one-dimensional kinetic Ising model are given by the functions $\phi(t)$, or equivalently the suscepti-

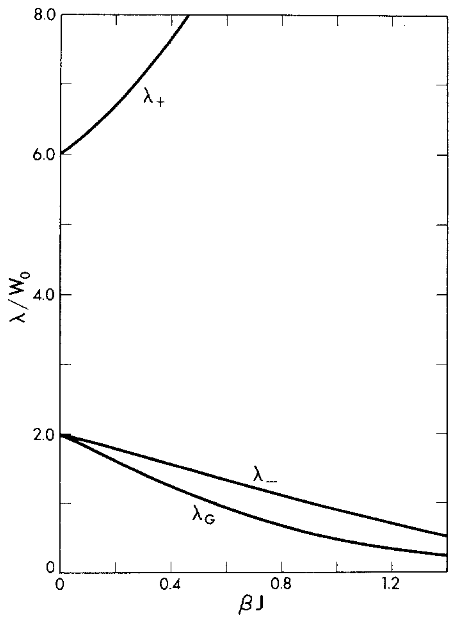

Fig. 1. The transient and slow relaxation rates (λ_+ and λ_-) of the new model compared with the single relaxation rate λ_G of the Glauber model as functions of βJ.

bilities $\chi(\omega)$. A few example situations will be considered here to make the results more tangible.

First, note that the response functions which characterize the two versions of the kinetic Ising model are in some ways similar. In both cases, $\phi(t)$ describes a monotonic relaxation toward equilibrium. This feature is common to all systems described by an essentially Hermitian master equation.[20,21]

There are also some important differences between the two versions of the kinetic Ising model. The most obvious difference is that $\phi(t)$ of the Glauber model is characterized by a single exponential decay rate λ_G. The modified model response function is characterized by both "transient" and "slow" decay rates λ_+ and λ_-.

One can get an intuitive picture of the dynamics of the two systems by comparing λ_G with λ_+ and λ_- as functions of βJ. Such a comparison is shown in Fig. 1 with $W_0 = 1$ for both models. One can see from this figure that the slow decay rate (λ_-) is comparable with λ_G, but the transient decay rate is always considerably larger. Both systems exhibit slowing down for low temperatures,

$$\lambda_G \to 4W_0 \exp(-4\beta J), \qquad \lambda_- \to 12W_0 \exp(-4\beta J)$$

This means that at low temperatures, the Glauber version of the kinetic Ising model is three times as slow as the new version, provided both models have the same value of W_0.

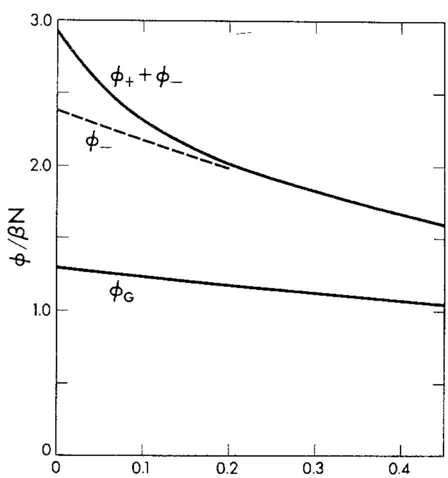

Fig. 2. The linear response functions $\phi(t)$ for the new model and the Glauber model. The dashed line shows the relaxation of the new model when the transient response is ignored.

The transient decay rate λ_+ is also shown in Fig. 1, and λ_+ increases rapidly with βJ so that in the low-temperature limit

$$\lambda_+ \to W_0 \exp(+4\beta J)$$

This rapid decay rate can be identified with the very rapid spin-flip process which aligns a spin with its parallel nearest neighbors. At higher temperatures, one cannot easily associate different spin-flip processes with the decay rates λ_+ and λ_-.

The transient and slow response of $\phi(t)$ can be easily seen in Fig. 2, where it is compared with $\phi_G(t)$ at an intermediate temperature ($\beta J = 1/2$).

The susceptibilities of the two versions of the kinetic Ising model presented here can be obtained exactly only because of their relative simplicity. In general, relaxation to equilibrium is characterized by an infinite spectrum of decay rates rather than a sum of one or two exponential decays.

APPENDIX

The following is a brief description of the steps necessary to transform T given by Eqs. (3.2) and (3.3) into the more convenient sum described by Eqs. (3.4)–(3.8).

Manipulation of the Pauli matrices of Eqs. (3.2)–(3.3) yields

$$T = \sum_n [(\overline{W} + W_0) + (\overline{W} - W_0)\sigma_{n-1}^z\sigma_{n+1}^z]\mathscr{T}_n \tag{A1}$$

with

$$\mathscr{T}_n = -\tfrac{1}{2}\sigma_n^x + \tfrac{1}{2}[\cosh\beta J - (\sinh\beta J)\sigma_n^z\sigma_{n+1}^z] \times [\cosh\beta J - (\sinh\beta J)\sigma_n^z\sigma_{n-1}^z] \tag{A2}$$

Further manipulation gives

$$\mathscr{T}_n = A_n^{-1}\left(\frac{1-\sigma_r^x}{2}\right)A_n^{-1} \tag{A3}$$

where

$$A_n^{-1} = [\cosh\tfrac{1}{2}\beta J - (\sinh\tfrac{1}{2}\beta J)\sigma_n^z\sigma_{n+1}^z] \times [\cosh\tfrac{1}{2}\beta J - (\sinh\tfrac{1}{2}\beta J)\sigma_n^z\sigma_{n-1}^z] \tag{A4}$$

and A_n is given by the above expression except that the two minus signs are changed to plus signs.

A set of projection operators G_n are defined as

$$G_n = A_n\left(\frac{1-\sigma_n^x}{2}\right)A_n^{-1} \tag{A5}$$

Then

$$T = \sum_n T_n \tag{A6}$$

with

$$T_n = [(\overline{W} + W_0) + (\overline{W} - W_0)\sigma_{n-1}^z\sigma_{n+1}^z]A_n^{-2}G_n \tag{A7}$$

Multiplication of the prefactor times A_n^{-2} gives

$$\begin{aligned} T_n = \{&[\overline{W}\cosh(2\beta J) + W_0] - \overline{W}(\sinh 2\beta J)\sigma_n^z(\sigma_{n+1}^z + \sigma_{n-1}^z) \\ &+ [\overline{W}\cosh(2\beta J) - W_0]\sigma_{n-1}^z\sigma_{n+1}^z\}G_n \end{aligned} \tag{A8}$$

This form for T is particularly convenient because its ground state $|\psi\rangle$ can be written as

$$|\psi\rangle = (2^N/Z)^{1/2}\prod_n \{[\cosh(\tfrac{1}{2}\beta J) + \sinh(\tfrac{1}{2}\beta J)]\sigma_n^z\sigma_{n+1}^z\}|0\rangle \tag{A9}$$

where $|0\rangle$ is the ground state of T when $\beta J = 0$. One can prove that $|\psi\rangle$ is the ground state of T. Let

$$|\psi_n\rangle = A_n^{-1}|\psi\rangle \tag{A10}$$

This new vector has spin n decoupled from all the other spins. That is, $|\psi_n\rangle$ can be separated into a sum of equal parts of spin up and spin down for spin n, times a function of all the other spins. This means that

$$G_n|\psi\rangle = A_n\left(\frac{1 - \sigma_n^x}{2}\right)|\psi_n\rangle = 0 \tag{A11}$$

and

$$T|\psi\rangle = 0 \tag{A12}$$

Similarly, since σ_n^z commutes with A_n^{-1}, and since

$$\left(\frac{1 - \sigma_n^x}{2}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{A13}$$

we have

$$G_n\sigma_n^z|\psi\rangle = \sigma_n^z|\psi\rangle \tag{A14}$$

The results of Eqs. (A5), (A6), (A8), (A11), and (A14) yield the properties of T described in Eqs. (3.4)–(3.8).

ACKNOWLEDGMENT

It is a pleasure to thank Prof. L. M. Falicov for his hospitality during my visit to the University of California, where this work was done.

REFERENCES

1. R. J. Glauber, *J. Math. Phys.* **4**:294 (1963).
2. R. Kikuchi, *Ann. Phys. (N.Y.)* **10**:127 (1960), **11**:306 (1960); Ya. Yu. Gotlib, *Sov. Phys.—Solid State* **3**:1577 (1962).

3. K. Kawasaki, in *Phase Transitions and Critical Phenomena, Vol. 2,* C. Domb and M. S. Green, eds. (Academic Press, New York, 1972).
4. K. Binder, in *Phase Transitions and Critical Phenomena*, Vol. 5b, C. Domb and M. S. Green, eds. (Academic Press, New York, 1972).
5. K. Kawasaki, *Phys. Rev.* **145**:224 (1966); **148**:375 (1966); **150**:285 (1966).
6. H. Yahata and M. Suzuki, *J. Phys. Soc. Japan* **27**:1421 (1969).
7. H. Yahata, *J. Phys. Soc. Japan* **30**:657 (1971).
8. A. Baumgartner and K. Binder, *J. Stat. Phys.* **18**:423 (1978).
9. H. J. Hilhorst, *Physica* **A79**:171 (1975).
10. E. Stoll, K. Binder, and T. Schneider, *Phys. Rev. B* **8**:3266 (1973).
11. M. Suzuki, *Int. J. Magn.* **1**:123 (1971).
12. Z. Racz and M. F. Collins, *Phys. Rev. B* **13**:3074 (1976).
13. N. J. White, *J. Phys. C* **9**:L187 (1976).
14. Z. C. Sepes and Z. Racz, *J. Phys. A* **11**:575 (1978).
15. M. Suzuki and R. Kubo, *J. Phys. Soc. Japan* **24**:51 (1968).
16. W. Kinzel, *Z. Phys. B* **29**:361 (1978).
17. Y. Achiam, *J. Phys. A* **11**:975 (1978).
18. L. P. Kadanoff and J. Swift, *Phys. Rev.* **165**:310 (1968).
19. R. Kubo, *J. Phys. Soc. Japan* **12**:570 (1957).
20. B. I. Halperin, *Phys. Rev. B* **8**:4437 (1973).
21. P. C. Hohenberg and B. I. Halperin, *Rev. Mod. Phys.* **49**:435 (1977).